Instanton counting and Donaldson invariants

Hardy Lecture, London

2010/07/01

Joint work with Lothar Göttsche, Kota Yoshioka,
relying on the work of Takuro Mochizuki

www.kurims.kyoto-u.ac.jp/~nakajima/Talks/2010-07-02 London.pdf

In this lecture,

X : compact, oriented, C^∞ 4-mfd with $b^+ > 1$, $b_1 = 0$

Let $\chi(X) = $ Euler #, $\sigma(X) = $ signature.

For convieniece, let

$$(K_X^2) := 2\chi(X) + 3\sigma(X) , \quad \chi_h(X) := \frac{\chi(X) + \sigma(X)}{4}$$

If X : complex projective surface, $\Longrightarrow \begin{cases} (K_X^2) = \text{self-intersection number of } K_X \\ \chi_h(X) = \text{holomorphic Euler characteristic} \end{cases}$

These are classical invariants of X.

We have two gauge theoretic invariants of X, defined as

— Take a Riemannian metric g on X.

— Consider the space of (equivalence classes of) solutions of nonlinear partial differential equations,

—— moduli spaces.

— Integrate certain natural cohomology classes over moduli spaces.

Donaldson invariants (1989)

······ an infinite sequence of C^∞-invariants of X, defined via moduli spaces of SO(3)-instantons

Seiberg-Witten invariants (1994)

······ an invariant of a spin^c-structure on X (zero except finitely many spin^c-structures), defined via moduli spaces of U(1)-monopoles

<u>Witten conjecture</u> (1994)

generating function of Donaldson invariants

= generating function of SW-invariants

Under a technical assumption : X: simple type

$$\mathcal{D}^{\mathfrak{z}}(\exp\alpha(1+\tfrac{1}{2}p)) = (-1)^{\chi_h(X)} 2^{(K_X^2)-\chi_h(X)+2} e^{(\alpha^2)/2} \sum_{\substack{\mathfrak{s}:\,\mathrm{spin}^c\,\mathrm{str} \\ (\text{finite sum})}} SW(\mathfrak{s})(-1)^{(\mathfrak{z},\mathfrak{z}+c_1(\mathfrak{s}))/2} e^{(c_1(\mathfrak{s}),\alpha)}$$

$\mathfrak{z} \in H^2(X;\mathbb{Z})$ (fixed)

$\alpha \in H_2(X;\mathbb{R})$, $p =$ pt class $\in H_0(X;\mathbb{R})$

I do not explain details of this formula. But it is striking:

- Moduli spaces of instantons and monopoles are close cousins, but no direct relations.

— In fact, monopoles are much easier to deal with, than instantons

- In LHS, infinitely many SO(3)-instanton moduli spaces with various p_1 are used.

$$\mathcal{D}^{\xi}(\exp(\alpha z)(1+\tfrac{1}{2}p)) = \sum_{n} \int_{M(2,3,n)} \exp(\mu(\alpha))(1+\tfrac{1}{2}\mu(p))$$

formal infinite sum

The above suggests $\exists$ structure on this family of infinitely many spaces.

Witten's argument was based on Seiberg-Witten ansatz for the $N=2$ SUSY Yang-Mills theory.

<u>Witten</u> (1988)

partition function = Donaldson invariants

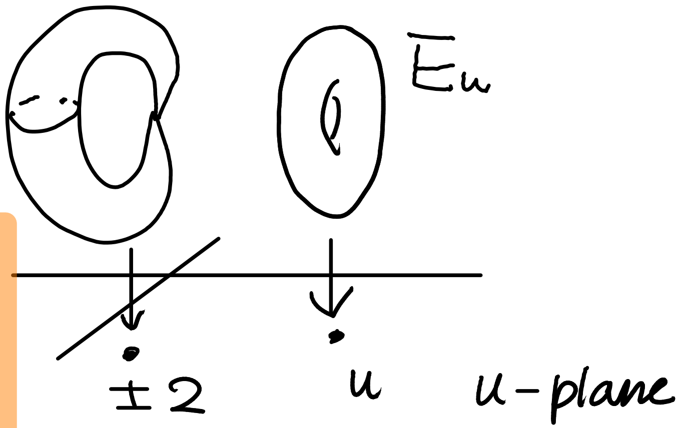
path integral over $\mathcal{B}$: the space of all connections + various fields

<u>SW</u> (1994)

This theory (for $X=\mathbb{R}^4$) is "controlled" by a family of elliptic curves :

$E_u : y^2 = 4x(x^2 + ux + 1)$

singular at $u = \pm 2$



$\left.\begin{array}{l}\text{Donaldson invariants} \\ \text{and SW invariants}\end{array}\right.$ appear as semiclassical expansions of the same quantum field theory at different points $\left\{\begin{array}{l} u = \infty \\ \text{and } u = \pm 2 \end{array}\right.$

$\frac{1}{u}$, $u \mp 2$: planck constants

At the present time, no one can justify this in a mathematically rigorous way.
But we <u>do</u> know answers to several natural questions :

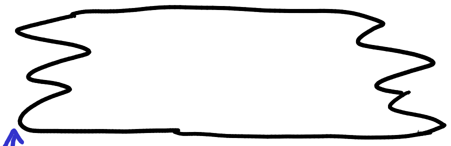
1) How does the partition function defined for $X = \mathbb{R}^4$?
— Nekrasov (2002) gave a rigorous definition,
using the equivariant homology group.

2) Why it is something to do with elliptic curves ?
— Nekrasov conjecture proved by
Nekrasov-Okounkov, N-Yoshioka, Braverman-Etingof.
This is a kind of "mirror symmetry".
The proofs are "computation".
..... Not so satisfactory, but at least rigorous .....

$\left(u = \right.$ generating function of certain equivariant integrals over
(framed) moduli space of $SO(3)$-instantons on $\left. \mathbb{R}^4 \right)$

The remaining question : How to apply our knowledge for $X = \mathbb{R}^4$
to general X ?
$\longrightarrow$ answered today.

Pidstrigach–Tyurin, Feehan–Leness proposed a rigorous approach using moduli spaces of SO(3)-monopoles as a cobordism between



U(1)-monopole moduli and SO(3)-instanton moduli.

Thus they connected SW and Donaldson invariants in a classical field theory.

Application

[FL] showed (under a certain technical assumption)

① $$D^{\mathfrak{z}}\left(\exp(\alpha z)\left(1+\tfrac{1}{2}p\right)\right) = \sum_{\mathfrak{s}} f\left(\chi_h(X), (K_X^2), \mathfrak{s}, \mathfrak{z}, \alpha, \mathfrak{s}_0\right) \times SW(\mathfrak{s})$$

auxiliary spin^c str

② Witten's conjecture is true if $(K_X^2) \geq \chi_h(X) - 3$ or X : projective surface

The coefficients f, defined via SO(3)-monopole moduli spaces,
are difficult to compute.

Also the role of the SW curve $y^2 = 4x(x^2+ux+1)$ was not clear in this approach.

Mochizuki (2009, preliminary version exists from 2002)
Assume X: cpx projective surface

① Replace moduli spaces of SO(3)-monopoles by their algebro-geometric counter parts:
moduli space of pairs of torsion free sheaves and their sections

② Define virtual fundamental classes on them
(like the case of Gromov-Witten invariants)

③ Apply virtual fixed point formula:
The coefficients f are now replaced by
explicit integrals over Hilbert scheme of points on X
----- nice resolution of $S^n X = X^n / \mathfrak{S}_n$

But still need to identify with those in Witten's conjecture.

GNY : Computation of the integral.

1°. enough to compute for X : toric surface

..... not trivial, but well-known since the work of Ellingsrud-Göttsche-Lehn

2°. fixed point formula $\Longrightarrow$ enough to compute for "$X = \mathbb{R}^4$"!

Th. ① Suppose X : cpx projective

$$\Longrightarrow \mathcal{D}^{3}\left(\exp \alpha \left(1+\frac{p}{2}\right)\right) = \sum_{\$} SW(\$) \operatorname*{Res}_{a=\infty} \mathcal{B}(\$, \mathfrak{z}; a)\, da$$

$\mathcal{B}(\$, \mathfrak{z} ; a)$: explicitly given in terms of an integral over (framed) moduli space of $SO(3)$-instantons on $X = \mathbb{R}^4$.

$$\sum_{n} \int_{M(2,n)} c_t(\operatorname{Ker} \mathcal{D}_A)$$

vector bdle of rank $= n$

Although the space is noncompact, this integral can be defined via equivariant homology groups

$N=2$ SUSY YM theory with a fundamental matter (Nekrasov)

Remark.

This formula makes sense even for X: C^∞ 4-mfd.

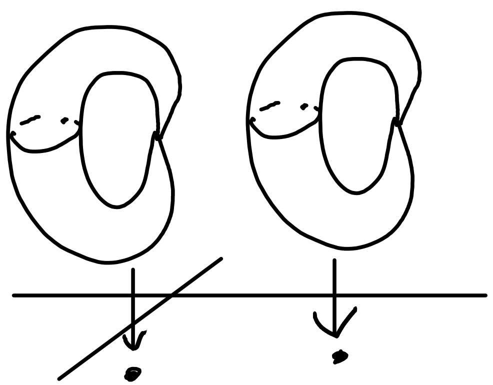
[Conjecture 1) is true also for C^∞ 4-mfd X.
Assume conjecture:

② $\mathfrak{B}(\mathfrak{s}, \mathfrak{z}; a)\,da$ can be written in terms of a family of singular elliptic curves.

SW curve for the theory with matter:

$$y^2 = 4x^2(x+u) + 4mx + 1 \qquad \text{where} \quad u \approx a^2$$

And the additional parameter $m \approx 1/t$ & t in $C_t(K_a D_A)$ is specialised so that it is a family of degenerate elliptic curves.



We get a different SW curve, since we have studied $SO(3)$-monopoles.

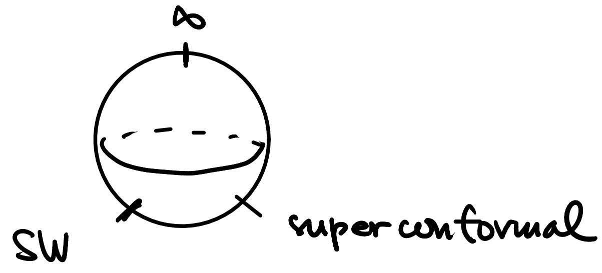
Then the remaining task is just a computation.

③ $\mathcal{B}(\$, \xi : a)\,da$; (after change of variable $a \to \phi^4$) extends to a meromorphic differential defined over $\mathbb{P}^1$

So far, it is close to Witten's intuition. But now we get a new feature :

— It has 3 poles

a) $\phi^4 = \infty$, b) SW contribution



and c) superconformal point [Mariño - Moore - Peradze]

($\Leftrightarrow$ both A & B cycles collapse)

So by Residue thm $\Rightarrow$ $\underset{\phi^4=\infty}{\mathrm{Res}} + \underset{SW}{\mathrm{Res}} + \underset{s.c.\,pt}{\mathrm{Res}} = 0$

$\Swarrow$ $\mathcal{D}$ by Th ①

$\Downarrow$ The RHS of Witten's conjecture

$\Downarrow$ a possible new contribution ?

Ref (Mariño-Moore-Peradze)

Assume X : SW simple type. We say X is of superconformal simple type

$\overset{\Longleftrightarrow}{\text{def.}}$ a) $(K_X^2) \geq \chi_h(X) - 3$

or b) $\displaystyle\sum_{\mathfrak{s}} (-1)^{(K_X, K_X + c_1(\mathfrak{s}))/2} SW(\mathfrak{s}) (c_1(\mathfrak{s}), \alpha)^n = 0$

$\forall\, 0 \leq n \leq \chi_h(X) - (K_X^2) - 4$

(Thm cont'd)

obviously true

④ Donaldson inv. depends only on $\mathfrak{z}$ mod 2 (up to sign)

$\Longrightarrow$ X : superconformal simple type.

$\Longrightarrow \displaystyle\sum_{\mathfrak{s}} SW(\mathfrak{s}) \mathfrak{B}(\mathfrak{s}, \mathfrak{z} ; a) da$ is regular at superconformal pt.

⑤ Residue Thm $\Longrightarrow$ Witten's conjecture is true.